FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

Version 0
September 17, 2008

STOP!!!!

Did you notice the FREEDOMS we've granted you?

Our copyright license is **different!** It grants freedoms rather than just imposing restrictions like all those other textbooks you probably own or use.

- We know people copy textbooks illegally but we would LOVE it if you copied our's - go ahead copy to your hearts content, **legally!**
- Publishers revenue is generated by controlling the market, we don't want any money, go ahead, distribute our books far and wide - we DARE you!
- Ever wanted to change your textbook? Of course you have! Go ahead change ours, make your own version, get your friends together, rip it apart and put it back together the way you like it. That's what we really want!
- Copy, modify, adapt, enhance, share, critique, adore, and contextualise. Do it all, do it with your colleagues, your friends or alone but get involved! Together we can overcome the challenges our complex and diverse country presents.
- So what is the catch? The only thing you can't do is take this book, make a few changes and then tell others that they can't do the same with your changes. It's share and share-alike and we know you'll agree that is only fair.
- These books were written by volunteers who want to help support education, who want the facts to be freely available for teachers to copy, adapt and re-use. Thousands of hours went into making them and they are a gift to everyone in the education community.

FHSST Core Team

Mark Horner ; Samuel Halliday ; Sarah Blyth ; Rory Adams ; Spencer Wheaton

FHSST Editors

Jaynie Padayachee ; Joanne Boulle ; Diana Mulcahy ; Annette Nell ; René Toerien ; Donovan Whitfield

FHSST Contributors

Rory Adams ; Prashant Arora ; Richard Baxter ; Dr. Sarah Blyth ; Sebastian Bodenstein ; Graeme Broster ; Richard Case ; Brett Cocks ; Tim Crombie ; Dr. Anne Dabrowski ; Laura Daniels ; Sean Dobbs ; Fernando Durrell ; Dr. Dan Dwyer ; Frans van Eeden ; Giovanni Franzoni ; Ingrid von Glehn ; Tamara von Glehn ; Lindsay Glesener ; Dr. Vanessa Godfrey ; Dr. Johan Gonzalez ; Hemant Gopal ; Umeshree Govender ; Heather Gray ; Lynn Greeff ; Dr. Tom Gutierrez ; Brooke Haag ; Kate Hadley ; Dr. Sam Halliday ; Asheena Hanuman ; Neil Hart ; Nicholas Hatcher ; Dr. Mark Horner ; Mfandaidza Hove ; Robert Hovden ; Jennifer Hsieh ; Clare Johnson ; Luke Jordan ; Tana Joseph ; Dr. Jennifer Klay ; Lara Kruger ; Sihle Kubheka ; Andrew Kubik ; Dr. Marco van Leeuwen ; Dr. Anton Machacek ; Dr. Komal Maheshwari ; Kosma von Maltitz ; Nicole Masureik ; John Mathew ; JoEllen McBride ; Nikolai Meures ; Riana Meyer ; Jenny Miller ; Abdul Mirza ; Asogan Moodaly ; Jothi Moodley ; Nolene Naidu ; Tyrone Negus ; Thomas O'Donnell ; Dr. Markus Oldenburg ; Dr. Jaynie Padayachee ; Nicolette Pekeur ; Sirika Pillay ; Jacques Plaut ; Andrea Prinsloo ; Joseph Raimondo ; Sanya Rajani ; Prof. Sergey Rakityansky ; Alastair Ramlakan ; Razvan Remsing ; Max Richter ; Sean Riddle ; Evan Robinson ; Dr. Andrew Rose ; Bianca Ruddy ; Katie Russell ; Duncan Scott ; Helen Seals ; Ian Sherratt ; Roger Sieloff ; Bradley Smith ; Greg Solomon ; Mike Stringer ; Shen Tian ; Robert Torregrosa ; Jimmy Tseng ; Helen Waugh ; Dr. Dawn Webber ; Michelle Wen ; Dr. Alexander Wetzler ; Dr. Spencer Wheaton ; Vivian White ; Dr. Gerald Wigger ; Harry Wiggins ; Wendy Williams ; Julie Wilson ; Andrew Wood ; Emma Wormauld ; Sahal Yacoob ; Jean Youssef

Contributors and editors have made a sincere effort to produce an accurate and useful resource. Should you have suggestions, find mistakes or be prepared to donate material for inclusion, please don't hesitate to contact us. We intend to work with all who are willing to help make this a continuously evolving resource!

www.fhsst.org

Contents

I Basics **1**

1 Introduction to Book **3**

1.1 The Language of Mathematics . . . 3

II Grade 10 **5**

2 Review of Past Work **7**

2.1 Introduction . . . 7
2.2 What is a number? . . . 7
2.3 Sets . . . 7
2.4 Letters and Arithmetic . . . 8
2.5 Addition and Subtraction . . . 9
2.6 Multiplication and Division . . . 9
2.7 Brackets . . . 9
2.8 Negative Numbers . . . 10
2.8.1 What is a negative number? . . . 10
2.8.2 Working with Negative Numbers . . . 11
2.8.3 Living Without the Number Line . . . 12
2.9 Rearranging Equations . . . 13
2.10 Fractions and Decimal Numbers . . . 15
2.11 Scientific Notation . . . 16
2.12 Real Numbers . . . 16
2.12.1 Natural Numbers . . . 17
2.12.2 Integers . . . 17
2.12.3 Rational Numbers . . . 17
2.12.4 Irrational Numbers . . . 19
2.13 Mathematical Symbols . . . 20
2.14 Infinity . . . 20
2.15 End of Chapter Exercises . . . 21

3 Rational Numbers - Grade 10 **23**

3.1 Introduction . . . 23
3.2 The Big Picture of Numbers . . . 23
3.3 Definition . . . 23

3.4 Forms of Rational Numbers 24
3.5 Converting Terminating Decimals into Rational Numbers 25
3.6 Converting Repeating Decimals into Rational Numbers 25
3.7 Summary 26
3.8 End of Chapter Exercises 27

4 Exponentials - Grade 10 **29**
4.1 Introduction 29
4.2 Definition 29
4.3 Laws of Exponents 30
4.3.1 Exponential Law 1: $a^0 = 1$ 30
4.3.2 Exponential Law 2: $a^m \times a^n = a^{m+n}$ 30
4.3.3 Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$ 31
4.3.4 Exponential Law 4: $a^m \div a^n = a^{m-n}$ 32
4.3.5 Exponential Law 5: $(ab)^n = a^n b^n$ 32
4.3.6 Exponential Law 6: $(a^m)^n = a^{mn}$ 33
4.4 End of Chapter Exercises 34

5 Estimating Surds - Grade 10 **37**
5.1 Introduction 37
5.2 Drawing Surds on the Number Line (Optional) 38
5.3 End of Chapter Excercises 39

6 Irrational Numbers and Rounding Off - Grade 10 **41**
6.1 Introduction 41
6.2 Irrational Numbers 41
6.3 Rounding Off 42
6.4 End of Chapter Exercises 43

7 Number Patterns - Grade 10 **45**
7.1 Common Number Patterns 45
7.1.1 Special Sequences 46
7.2 Make your own Number Patterns 46
7.3 Notation 47
7.3.1 Patterns and Conjecture 49
7.4 Exercises 50

8 Finance - Grade 10 **53**
8.1 Introduction 53
8.2 Foreign Exchange Rates 53
8.2.1 How much is R1 really worth? 53
8.2.2 Cross Currency Exchange Rates 56
8.2.3 Enrichment: Fluctuating exchange rates 57
8.3 Being Interested in Interest 58

8.4 Simple Interest . . . 59
8.4.1 Other Applications of the Simple Interest Formula . . . 61
8.5 Compound Interest . . . 63
8.5.1 Fractions add up to the Whole . . . 65
8.5.2 The Power of Compound Interest . . . 65
8.5.3 Other Applications of Compound Growth . . . 67
8.6 Summary . . . 68
8.6.1 Definitions . . . 68
8.6.2 Equations . . . 68
8.7 End of Chapter Exercises . . . 69

9 Products and Factors - Grade 10 71

9.1 Introduction . . . 71
9.2 Recap of Earlier Work . . . 71
9.2.1 Parts of an Expression . . . 71
9.2.2 Product of Two Binomials . . . 71
9.2.3 Factorisation . . . 72
9.3 More Products . . . 74
9.4 Factorising a Quadratic . . . 76
9.5 Factorisation by Grouping . . . 79
9.6 Simplification of Fractions . . . 80
9.7 End of Chapter Exercises . . . 82

10 Equations and Inequalities - Grade 10 83

10.1 Strategy for Solving Equations . . . 83
10.2 Solving Linear Equations . . . 84
10.3 Solving Quadratic Equations . . . 89
10.4 Exponential Equations of the form $ka^{(x+p)} = m$ . . . 93
10.4.1 Algebraic Solution . . . 93
10.5 Linear Inequalities . . . 96
10.6 Linear Simultaneous Equations . . . 99
10.6.1 Finding solutions . . . 99
10.6.2 Graphical Solution . . . 99
10.6.3 Solution by Substitution . . . 101
10.7 Mathematical Models . . . 103
10.7.1 Introduction . . . 103
10.7.2 Problem Solving Strategy . . . 104
10.7.3 Application of Mathematical Modelling . . . 104
10.7.4 End of Chapter Exercises . . . 106
10.8 Introduction to Functions and Graphs . . . 107
10.9 Functions and Graphs in the Real-World . . . 107
10.10Recap . . . 107

10.10.1 Variables and Constants . . . 107

10.10.2 Relations and Functions . . . 108

10.10.3 The Cartesian Plane . . . 108

10.10.4 Drawing Graphs . . . 109

10.10.5 Notation used for Functions . . . 110

10.11 Characteristics of Functions - All Grades . . . 112

10.11.1 Dependent and Independent Variables . . . 112

10.11.2 Domain and Range . . . 113

10.11.3 Intercepts with the Axes . . . 113

10.11.4 Turning Points . . . 114

10.11.5 Asymptotes . . . 114

10.11.6 Lines of Symmetry . . . 114

10.11.7 Intervals on which the Function Increases/Decreases . . . 114

10.11.8 Discrete or Continuous Nature of the Graph . . . 114

10.12 Graphs of Functions . . . 116

10.12.1 Functions of the form $y = ax + q$ . . . 116

10.12.2 Functions of the Form $y = ax^2 + q$ . . . 120

10.12.3 Functions of the Form $y = \frac{a}{x} + q$ . . . 125

10.12.4 Functions of the Form $y = ab^{(x)} + q$ . . . 129

10.13 End of Chapter Exercises . . . 133

11 Average Gradient - Grade 10 Extension 135

11.1 Introduction . . . 135

11.2 Straight-Line Functions . . . 135

11.3 Parabolic Functions . . . 136

11.4 End of Chapter Exercises . . . 138

12 Geometry Basics 139

12.1 Introduction . . . 139

12.2 Points and Lines . . . 139

12.3 Angles . . . 140

12.3.1 Measuring angles . . . 141

12.3.2 Special Angles . . . 141

12.3.3 Special Angle Pairs . . . 143

12.3.4 Parallel Lines intersected by Transversal Lines . . . 143

12.4 Polygons . . . 147

12.4.1 Triangles . . . 147

12.4.2 Quadrilaterals . . . 152

12.4.3 Other polygons . . . 155

12.4.4 Extra . . . 156

12.5 Exercises . . . 157

12.5.1 Challenge Problem . . . 159

13 Geometry - Grade 10 **161**

13.1 Introduction . . . 161

13.2 Right Prisms and Cylinders . . . 161

13.2.1 Surface Area . . . 162

13.2.2 Volume . . . 164

13.3 Polygons . . . 167

13.3.1 Similarity of Polygons . . . 167

13.4 Co-ordinate Geometry . . . 171

13.4.1 Introduction . . . 171

13.4.2 Distance between Two Points . . . 172

13.4.3 Calculation of the Gradient of a Line . . . 173

13.4.4 Midpoint of a Line . . . 174

13.5 Transformations . . . 177

13.5.1 Translation of a Point . . . 177

13.5.2 Reflection of a Point . . . 179

13.6 End of Chapter Exercises . . . 185

14 Trigonometry - Grade 10 **189**

14.1 Introduction . . . 189

14.2 Where Trigonometry is Used . . . 190

14.3 Similarity of Triangles . . . 190

14.4 Definition of the Trigonometric Functions . . . 191

14.5 Simple Applications of Trigonometric Functions . . . 195

14.5.1 Height and Depth . . . 195

14.5.2 Maps and Plans . . . 197

14.6 Graphs of Trigonometric Functions . . . 199

14.6.1 Graph of $\sin\theta$ . . . 199

14.6.2 Functions of the form $y = a\sin(x) + q$ . . . 200

14.6.3 Graph of $\cos\theta$ . . . 202

14.6.4 Functions of the form $y = a\cos(x) + q$ . . . 202

14.6.5 Comparison of Graphs of $\sin\theta$ and $\cos\theta$ . . . 204

14.6.6 Graph of $\tan\theta$ . . . 204

14.6.7 Functions of the form $y = a\tan(x) + q$ . . . 205

14.7 End of Chapter Exercises . . . 208

15 Statistics - Grade 10 **211**

15.1 Introduction . . . 211

15.2 Recap of Earlier Work . . . 211

15.2.1 Data and Data Collection . . . 211

15.2.2 Methods of Data Collection . . . 212

15.2.3 Samples and Populations . . . 213

15.3 Example Data Sets . . . 213

15.3.1 Data Set 1: Tossing a Coin 213
15.3.2 Data Set 2: Casting a die 213
15.3.3 Data Set 3: Mass of a Loaf of Bread 214
15.3.4 Data Set 4: Global Temperature 214
15.3.5 Data Set 5: Price of Petrol 215
15.4 Grouping Data 215
15.4.1 Exercises - Grouping Data 216
15.5 Graphical Representation of Data 217
15.5.1 Bar and Compound Bar Graphs 217
15.5.2 Histograms and Frequency Polygons 217
15.5.3 Pie Charts 219
15.5.4 Line and Broken Line Graphs 220
15.5.5 Exercises - Graphical Representation of Data 221
15.6 Summarising Data 222
15.6.1 Measures of Central Tendency 222
15.6.2 Measures of Dispersion 225
15.6.3 Exercises - Summarising Data 228
15.7 Misuse of Statistics 229
15.7.1 Exercises - Misuse of Statistics 230
15.8 Summary of Definitions 232
15.9 Exercises 232

16 Probability - Grade 10 **235**

16.1 Introduction 235
16.2 Random Experiments 235
16.2.1 Sample Space of a Random Experiment 235
16.3 Probability Models 238
16.3.1 Classical Theory of Probability 239
16.4 Relative Frequency *vs.* Probability 240
16.5 Project Idea 242
16.6 Probability Identities 242
16.7 Mutually Exclusive Events 243
16.8 Complementary Events 244
16.9 End of Chapter Exercises 246

III Grade 11 **249**

17 Exponents - Grade 11 **251**

17.1 Introduction 251
17.2 Laws of Exponents 251
17.2.1 Exponential Law 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ 251
17.3 Exponentials in the Real-World 253
17.4 End of chapter Exercises 254

18 Surds - Grade 11 **255**

18.1 Surd Calculations . . . 255

18.1.1 Surd Law 1: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ . . . 255

18.1.2 Surd Law 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ . . . 255

18.1.3 Surd Law 3: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ . . . 256

18.1.4 Like and Unlike Surds . . . 256

18.1.5 Simplest Surd form . . . 257

18.1.6 Rationalising Denominators . . . 258

18.2 End of Chapter Exercises . . . 259

19 Error Margins - Grade 11 **261**

20 Quadratic Sequences - Grade 11 **265**

20.1 Introduction . . . 265

20.2 What is a *quadratic sequence*? . . . 265

20.3 End of chapter Exercises . . . 269

21 Finance - Grade 11 **271**

21.1 Introduction . . . 271

21.2 Depreciation . . . 271

21.3 Simple Depreciation (it really is simple!) . . . 271

21.4 Compound Depreciation . . . 274

21.5 Present Values or Future Values of an Investment or Loan . . . 276

21.5.1 Now or Later . . . 276

21.6 Finding i . . . 278

21.7 Finding n - Trial and Error . . . 279

21.8 Nominal and Effective Interest Rates . . . 280

21.8.1 The General Formula . . . 281

21.8.2 De-coding the Terminology . . . 282

21.9 Formulae Sheet . . . 284

21.9.1 Definitions . . . 284

21.9.2 Equations . . . 285

21.10End of Chapter Exercises . . . 285

22 Solving Quadratic Equations - Grade 11 **287**

22.1 Introduction . . . 287

22.2 Solution by Factorisation . . . 287

22.3 Solution by Completing the Square . . . 290

22.4 Solution by the Quadratic Formula . . . 293

22.5 Finding an equation when you know its roots . . . 296

22.6 End of Chapter Exercises . . . 299

23 Solving Quadratic Inequalities - Grade 11 **301**

23.1 Introduction . . . 301

23.2 Quadratic Inequalities . . . 301

23.3 End of Chapter Exercises . . . 304

24 Solving Simultaneous Equations - Grade 11 **307**

24.1 Graphical Solution . . . 307

24.2 Algebraic Solution . . . 309

25 Mathematical Models - Grade 11 **313**

25.1 Real-World Applications: Mathematical Models . . . 313

25.2 End of Chatpter Exercises . . . 317

26 Quadratic Functions and Graphs - Grade 11 **321**

26.1 Introduction . . . 321

26.2 Functions of the Form $y = a(x + p)^2 + q$ . . . 321

26.2.1 Domain and Range . . . 322

26.2.2 Intercepts . . . 323

26.2.3 Turning Points . . . 324

26.2.4 Axes of Symmetry . . . 325

26.2.5 Sketching Graphs of the Form $f(x) = a(x + p)^2 + q$ . . . 325

26.2.6 Writing an equation of a shifted parabola . . . 327

26.3 End of Chapter Exercises . . . 327

27 Hyperbolic Functions and Graphs - Grade 11 **329**

27.1 Introduction . . . 329

27.2 Functions of the Form $y = \frac{a}{x+p} + q$ . . . 329

27.2.1 Domain and Range . . . 330

27.2.2 Intercepts . . . 331

27.2.3 Asymptotes . . . 332

27.2.4 Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$ . . . 333

27.3 End of Chapter Exercises . . . 333

28 Exponential Functions and Graphs - Grade 11 **335**

28.1 Introduction . . . 335

28.2 Functions of the Form $y = ab^{(x+p)} + q$ . . . 335

28.2.1 Domain and Range . . . 336

28.2.2 Intercepts . . . 337

28.2.3 Asymptotes . . . 338

28.2.4 Sketching Graphs of the Form $f(x) = ab^{(x+p)} + q$ . . . 338

28.3 End of Chapter Exercises . . . 339

29 Gradient at a Point - Grade 11 **341**

29.1 Introduction . . . 341

29.2 Average Gradient . . . 341

29.3 End of Chapter Exercises . . . 344

30 Linear Programming - Grade 11 **345**
30.1 Introduction 345
30.2 Terminology 345
30.2.1 Decision Variables 345
30.2.2 Objective Function 345
30.2.3 Constraints 346
30.2.4 Feasible Region and Points 346
30.2.5 The Solution 346
30.3 Example of a Problem 347
30.4 Method of Linear Programming 347
30.5 Skills you will need 347
30.5.1 Writing Constraint Equations 347
30.5.2 Writing the Objective Function 348
30.5.3 Solving the Problem 350
30.6 End of Chapter Exercises 352

31 Geometry - Grade 11 **357**
31.1 Introduction 357
31.2 Right Pyramids, Right Cones and Spheres 357
31.3 Similarity of Polygons 360
31.4 Triangle Geometry 361
31.4.1 Proportion 361
31.5 Co-ordinate Geometry 368
31.5.1 Equation of a Line between Two Points 368
31.5.2 Equation of a Line through One Point and Parallel or Perpendicular to Another Line 371
31.5.3 Inclination of a Line 371
31.6 Transformations 373
31.6.1 Rotation of a Point 373
31.6.2 Enlargement of a Polygon 1 376

32 Trigonometry - Grade 11 **381**
32.1 History of Trigonometry 381
32.2 Graphs of Trigonometric Functions 381
32.2.1 Functions of the form $y = \sin(k\theta)$ 381
32.2.2 Functions of the form $y = \cos(k\theta)$ 383
32.2.3 Functions of the form $y = \tan(k\theta)$ 384
32.2.4 Functions of the form $y = \sin(\theta + p)$ 385
32.2.5 Functions of the form $y = \cos(\theta + p)$ 386
32.2.6 Functions of the form $y = \tan(\theta + p)$ 387
32.3 Trigonometric Identities 389
32.3.1 Deriving Values of Trigonometric Functions for $30°$, $45°$ and $60°$ 389
32.3.2 Alternate Definition for $\tan\theta$ 391

32.3.3 A Trigonometric Identity . . . 392
32.3.4 Reduction Formula . . . 394
32.4 Solving Trigonometric Equations . . . 399
32.4.1 Graphical Solution . . . 399
32.4.2 Algebraic Solution . . . 401
32.4.3 Solution using CAST diagrams . . . 403
32.4.4 General Solution Using Periodicity . . . 405
32.4.5 Linear Trigonometric Equations . . . 406
32.4.6 Quadratic and Higher Order Trigonometric Equations . . . 406
32.4.7 More Complex Trigonometric Equations . . . 407
32.5 Sine and Cosine Identities . . . 409
32.5.1 The Sine Rule . . . 409
32.5.2 The Cosine Rule . . . 412
32.5.3 The Area Rule . . . 414
32.6 Exercises . . . 416

33 Statistics - Grade 11 **419**

33.1 Introduction . . . 419
33.2 Standard Deviation and Variance . . . 419
33.2.1 Variance . . . 419
33.2.2 Standard Deviation . . . 421
33.2.3 Interpretation and Application . . . 423
33.2.4 Relationship between Standard Deviation and the Mean . . . 424
33.3 Graphical Representation of Measures of Central Tendency and Dispersion . . . 424
33.3.1 Five Number Summary . . . 424
33.3.2 Box and Whisker Diagrams . . . 425
33.3.3 Cumulative Histograms . . . 426
33.4 Distribution of Data . . . 428
33.4.1 Symmetric and Skewed Data . . . 428
33.4.2 Relationship of the Mean, Median, and Mode . . . 428
33.5 Scatter Plots . . . 429
33.6 Misuse of Statistics . . . 432
33.7 End of Chapter Exercises . . . 435

34 Independent and Dependent Events - Grade 11 **437**

34.1 Introduction . . . 437
34.2 Definitions . . . 437
34.2.1 Identification of Independent and Dependent Events . . . 438
34.3 End of Chapter Exercises . . . 441

IV Grade 12 **443**

35 Logarithms - Grade 12 **445**

35.1 Definition of Logarithms . . . 445

35.2 Logarithm Bases 446
35.3 Laws of Logarithms 447
35.4 Logarithm Law 1: $\log_a 1 = 0$ 447
35.5 Logarithm Law 2: $\log_a(a) = 1$ 448
35.6 Logarithm Law 3: $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$ 448
35.7 Logarithm Law 4: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ 449
35.8 Logarithm Law 5: $\log_a(x^b) = b\log_a(x)$ 450
35.9 Logarithm Law 6: $\log_a\left(\sqrt[b]{x}\right) = \frac{\log_a(x)}{b}$ 450
35.10 Solving simple log equations 452
35.10.1 Exercises 454
35.11 Logarithmic applications in the Real World 454
35.11.1 Exercises 455
35.12 End of Chapter Exercises 455

36 Sequences and Series - Grade 12 **457**
36.1 Introduction 457
36.2 Arithmetic Sequences 457
36.2.1 General Equation for the n^{th}-term of an Arithmetic Sequence 458
36.3 Geometric Sequences 459
36.3.1 Example - A Flu Epidemic 459
36.3.2 General Equation for the n^{th}-term of a Geometric Sequence 461
36.3.3 Exercises 461
36.4 Recursive Formulae for Sequences 462
36.5 Series 463
36.5.1 Some Basics 463
36.5.2 Sigma Notation 463
36.6 Finite Arithmetic Series 465
36.6.1 General Formula for a Finite Arithmetic Series 466
36.6.2 Exercises 467
36.7 Finite Squared Series 468
36.8 Finite Geometric Series 469
36.8.1 Exercises 470
36.9 Infinite Series 471
36.9.1 Infinite Geometric Series 471
36.9.2 Exercises 472
36.10 End of Chapter Exercises 472

37 Finance - Grade 12 **477**
37.1 Introduction 477
37.2 Finding the Length of the Investment or Loan 477
37.3 A Series of Payments 478
37.3.1 Sequences and Series 479

37.3.2 Present Values of a series of Payments 479
37.3.3 Future Value of a series of Payments 484
37.3.4 Exercises - Present and Future Values 485
37.4 Investments and Loans 485
37.4.1 Loan Schedules 485
37.4.2 Exercises - Investments and Loans 489
37.4.3 Calculating Capital Outstanding 489
37.5 Formulae Sheet 489
37.5.1 Definitions 490
37.5.2 Equations 490
37.6 End of Chapter Exercises 490

38 Factorising Cubic Polynomials - Grade 12 493
38.1 Introduction 493
38.2 The Factor Theorem 493
38.3 Factorisation of Cubic Polynomials 494
38.4 Exercises - Using Factor Theorem 496
38.5 Solving Cubic Equations 496
38.5.1 Exercises - Solving of Cubic Equations 498
38.6 End of Chapter Exercises 498

39 Functions and Graphs - Grade 12 501
39.1 Introduction 501
39.2 Definition of a Function 501
39.2.1 Exercises 501
39.3 Notation used for Functions 502
39.4 Graphs of Inverse Functions 502
39.4.1 Inverse Function of $y = ax + q$ 503
39.4.2 Exercises 504
39.4.3 Inverse Function of $y = ax^2$ 504
39.4.4 Exercises 504
39.4.5 Inverse Function of $y = a^x$ 506
39.4.6 Exercises 506
39.5 End of Chapter Exercises 507

40 Differential Calculus - Grade 12 509
40.1 Why do I have to learn this stuff? 509
40.2 Limits 510
40.2.1 A Tale of Achilles and the Tortoise 510
40.2.2 Sequences, Series and Functions 511
40.2.3 Limits 512
40.2.4 Average Gradient and Gradient at a Point 516
40.3 Differentiation from First Principles 519

40.4 Rules of Differentiation . . . 521
40.4.1 Summary of Differentiation Rules . . . 522
40.5 Applying Differentiation to Draw Graphs . . . 523
40.5.1 Finding Equations of Tangents to Curves . . . 523
40.5.2 Curve Sketching . . . 524
40.5.3 Local minimum, Local maximum and Point of Inflextion . . . 529
40.6 Using Differential Calculus to Solve Problems . . . 530
40.6.1 Rate of Change problems . . . 534
40.7 End of Chapter Exercises . . . 535

41 Linear Programming - Grade 12 **539**
41.1 Introduction . . . 539
41.2 Terminology . . . 539
41.2.1 Feasible Region and Points . . . 539
41.3 Linear Programming and the Feasible Region . . . 540
41.4 End of Chapter Exercises . . . 546

42 Geometry - Grade 12 **549**
42.1 Introduction . . . 549
42.2 Circle Geometry . . . 549
42.2.1 Terminology . . . 549
42.2.2 Axioms . . . 550
42.2.3 Theorems of the Geometry of Circles . . . 550
42.3 Co-ordinate Geometry . . . 566
42.3.1 Equation of a Circle . . . 566
42.3.2 Equation of a Tangent to a Circle at a Point on the Circle . . . 569
42.4 Transformations . . . 571
42.4.1 Rotation of a Point about an angle θ . . . 571
42.4.2 Characteristics of Transformations . . . 573
42.4.3 Characteristics of Transformations . . . 573
42.5 Exercises . . . 574

43 Trigonometry - Grade 12 **577**
43.1 Compound Angle Identities . . . 577
43.1.1 Derivation of $\sin(\alpha + \beta)$ . . . 577
43.1.2 Derivation of $\sin(\alpha - \beta)$ . . . 578
43.1.3 Derivation of $\cos(\alpha + \beta)$ . . . 578
43.1.4 Derivation of $\cos(\alpha - \beta)$ . . . 579
43.1.5 Derivation of $\sin 2\alpha$ . . . 579
43.1.6 Derivation of $\cos 2\alpha$ . . . 579
43.1.7 Problem-solving Strategy for Identities . . . 580
43.2 Applications of Trigonometric Functions . . . 582
43.2.1 Problems in Two Dimensions . . . 582

43.2.2 Problems in 3 dimensions 584
43.3 Other Geometries 586
43.3.1 Taxicab Geometry 586
43.3.2 Manhattan distance 586
43.3.3 Spherical Geometry 587
43.3.4 Fractal Geometry 588
43.4 End of Chapter Exercises 589

44 Statistics - Grade 12 591

44.1 Introduction 591
44.2 A Normal Distribution 591
44.3 Extracting a Sample Population 593
44.4 Function Fitting and Regression Analysis 594
44.4.1 The Method of Least Squares 596
44.4.2 Using a calculator 597
44.4.3 Correlation coefficients 599
44.5 Exercises 600

45 Combinations and Permutations - Grade 12 603

45.1 Introduction 603
45.2 Counting 603
45.2.1 Making a List 603
45.2.2 Tree Diagrams 604
45.3 Notation 604
45.3.1 The Factorial Notation 604
45.4 The Fundamental Counting Principle 604
45.5 Combinations 605
45.5.1 Counting Combinations 605
45.5.2 Combinatorics and Probability 606
45.6 Permutations 606
45.6.1 Counting Permutations 607
45.7 Applications 608
45.8 Exercises 610

V Exercises 613

46 General Exercises 615

47 Exercises - Not covered in Syllabus 617

A GNU Free Documentation License 619

Chapter 21

Finance - Grade 11

21.1 Introduction

In Grade 10, the ideas of simple and compound interest was introduced. In this chapter we will be extending those ideas, so it is a good idea to go back to Chapter 8 and revise what you learnt in Grade 10. If you master the techniques in this chapter, you will understand about depreciation and will learn how to determine which bank is offering the better interest rate.

21.2 Depreciation

It is said that when you drive a new car out of the dealership, it loses 20% of its value, because it is now "second-hand". And from there on the value keeps falling, or *depreciating*. Second hand cars are cheaper than new cars, and the older the car, usually the cheaper it is. If you buy a second hand (or should we say *pre-owned*!) car from a dealership, they will base the price on something called *book value*.

The book value of the car is the value of the car taking into account the loss in value due to wear, age and use. We call this loss in value *depreciation*, and in this section we will look at two ways of how this is calculated. Just like interest rates, the two methods of calculating depreciation are *simple* and *compound* methods.

The terminology used for simple depreciation is **straight-line depreciation** and for compound depreciation is **reducing-balance depreciation**. In the straight-line method the value of the asset is reduced by the same constant amount each year. In the compound depreciation method the value of the asset is reduced by the same percentage each year. This means that the value of an asset does not decrease by a constant amount each year, but the decrease is most in the first year, then by a smaller amount in the second year and by even a smaller amount in the third year, and so on.

Extension: Depreciation

You may be wondering why we need to calculate depreciation. Determining the value of assets (as in the example of the second hand cars) is one reason, but there is also a more financial reason for calculating depreciation - tax! Companies can take depreciation into account as an expense, and thereby reduce their taxable income. A lower taxable income means that the company will pay less income tax to the Revenue Service.

21.3 Simple Depreciation (it really is simple!)

Let us go back to the second hand cars. One way of calculating a depreciation amount would be to assume that the car has a limited useful life. Simple depreciation assumes that the value of the car decreases by an equal amount each year. For example, let us say the limited useful life of a car is 5 years, and the cost of the car today is R60 000. What we are saying is that after 5 years you will have to buy a new car, which means that the old one will be valueless at that point in time. Therefore, the amount of depreciation is calculated:

$$\frac{\text{R60 000}}{\text{5 years}} = \text{R12 000} \quad \text{per year.}$$

The value of the car is then:

End of Year 1	R60 000 - 1×(R12 000)	= R48 000
End of Year 2	R60 000 - 2×(R12 000)	= R36 000
End of Year 3	R60 000 - 3×(R12 000)	= R24 000
End of Year 4	R60 000 - 4×(R12 000)	= R12 000
End of Year 5	R60 000 - 5×(R12 000)	= R0

This looks similar to the formula for simple interest:

$$\text{Total Interest after } n \text{ years} = n \times (P \times i)$$

where i is the annual percentage interest rate and P is the principal amount.

If we replace the word *interest* with the word *depreciation* and the word *principal* with the words *initial value* we can use the same formula:

$$\text{Total depreciation after } n \text{ years} = n \times (P \times i)$$

Then the book value of the asset after n years is:

$$\begin{aligned}\text{Initial Value - Total depreciation after } n \text{ years} &= P - n \times (P \times i) \\ &= P(1 - n \times i)\end{aligned}$$

For example, the book value of the car after two years can be simply calculated as follows:

$$\begin{aligned}\text{Book Value after 2 years} &= P(1 - n \times i) \\ &= \text{R60 000}(1 - 2 \times 20\%) \\ &= \text{R60 000}(1 - 0{,}4) \\ &= \text{R60 000}(0{,}6) \\ &= \text{R36 000}\end{aligned}$$

as expected.

Note that the difference between the simple interest calculations and the simple depreciation calculations is that while the interest adds value to the principal amount, the depreciation amount reduces value!

Worked Example 96: Simple Depreciation method

Question: A car is worth $R240\ 000$ now. If it depreciates at a rate of 15% p.a. on a staight-line depreciation, what is it worth in 5 years' time ?

Answer

Step 1 : Determine what has been provided and what is required

$$\begin{aligned} P &= R240\ 000 \\ i &= 0{,}15 \\ n &= 5 \\ A & \ is \ required \end{aligned}$$

Step 2 : Determine how to approach the problem

$$A = 240\ 000(1 - 0{,}15 \times 5)$$

Step 3 : Solve the problem

$$\begin{aligned} A &= 240\ 000(1 - 0{,}75) \\ &= 240\ 000 \times 0{,}25 \\ &= 60\ 000 \end{aligned}$$

Step 4 : Write the final answer
In 5 years' time the car is worth $R60\ 000$

Worked Example 97: Simple Depreciation

Question: A small business buys a photocopier for R 12 000. For the tax return the owner depreciates this asset over 3 years using a straight-line depreciation method. What amount will he fill in on his tax form after 1 year, after 2 years and then after 3 years ?

Answer

Step 1 : Understanding the question
The owner of the business wants the photocopier to depreciates to R0 after 3 years. Thus, the value of the photocopier will go down by $12\ 000 \div 3 = R4\ 000$ per year.

Step 2 : Value of the photocopier after 1 year
$12\ 000 - 4\ 000 = R8\ 000$

Step 3 : Value of the machine after 2 years
$8\ 000 - 4\ 000 = R4\ 000$

Step 4 : Write the final answer
$4\ 000 - 4\ 000 = 0$
After 3 years the photocopier is worth nothing

Extension: Salvage Value

Looking at the same example of our car with an initial value of R60 000, what if we suppose that we think we would be able to sell the car at the end of the 5 year period for R10 000? We call this amount the "Salvage Value"

We are still assuming simple depreciation over a useful life of 5 years, but now instead of depreciating the full value of the asset, we will take into account the salvage value, and will only apply the depreciation to the value of the asset that we expect not to recoup, i.e. R60 000 - R10 000 = R50 000.

The annual depreciation amount is then calculated as (R60 000 - R10 000) / 5 = R10 000

In general, the for simple (straight line) depreciation:

$$\text{Annual Depreciation} = \frac{\text{Initial Value - Salvage Value}}{\text{Useful Life}}$$

?

Exercise: Simple Depreciation

1. A business buys a truck for R560 000. Over a period of 10 years the value of the truck depreciates to R0 (using the straight-line method). What is the value of the truck after 8 years ?
2. Shrek wants to buy his grandpa's donkey for R800. His grandpa is quite pleased with the offer, seeing that it only depreciated at a rate of 3% per year using the straight-line method. Grandpa bought the donkey 5 years ago. What did grandpa pay for the donkey then ?
3. Seven years ago, Rocco's drum kit cost him R 12 500. It has now been valued at R2 300. What rate of simple depreciation does this represent ?
4. Fiona buys a DsTV satellite dish for R3 000. Due to weathering, its value depreciates simply at 15% per annum. After how long will the satellite dish be worth nothing ?

21.4 Compound Depreciation

The second method of calculating depreciation is to assume that the value of the asset decreases at a certain annual rate, but that the initial value of the asset this year, is the book value of the asset at the end of last year.

For example, if our second hand car has a limited useful life of 5 years and it has an initial value of R60 000, then the interest rate of depreciation is 20% (100%/5 years). After 1 year, the car is worth:

$$\begin{aligned} \text{Book Value after first year} &= P(1 - n \times i) \\ &= \text{R60 000}(1 - 1 \times 20\%) \\ &= \text{R60 000}(1 - 0{,}2) \\ &= \text{R60 000}(0{,}8) \\ &= \text{R48 000} \end{aligned}$$

At the beginning of the second year, the car is now worth R48 000, so after two years, the car is worth:

$$\begin{aligned} \text{Book Value after second year} &= P(1 - n \times i) \\ &= \text{R48 000}(1 - 1 \times 20\%) \\ &= \text{R48 000}(1 - 0{,}2) \\ &= \text{R48 000}(0{,}8) \\ &= \text{R38 400} \end{aligned}$$

We can tabulate these values.

End of first year	R60 000$(1-1\times 20\%)$=R60 000$(1-1\times 20\%)^1$	= R48 000,00
End of second year	R48 000$(1-1\times 20\%)$=R60 000$(1-1\times 20\%)^2$	= R38 400,00
End of third year	R38 400$(1-1\times 20\%)$=R60 000$(1-1\times 20\%)^3$	= R30 720,00
End of fourth year	R30 720$(1-1\times 20\%)$=R60 000$(1-1\times 20\%)^4$	= R24 576,00
End of fifth year	R24 576$(1-1\times 20\%)$=R60 000$(1-1\times 20\%)^5$	= R19 608,80

We can now write a general formula for the book value of an asset if the depreciation is compounded.

$$\text{Initial Value - Total depreciation after } n \text{ years} = P(1-i)^n \tag{21.1}$$

For example, the book value of the car after two years can be simply calculated as follows:

$$\begin{aligned}\text{Book Value after 2 years} &= P(1-i)^n \\ &= \text{R60 000}(1-20\%)^2 \\ &= \text{R60 000}(1-0{,}2)^2 \\ &= \text{R60 000}(0{,}8)^2 \\ &= \text{R38 400}\end{aligned}$$

as expected.

Note that the difference between the compound interest calculations and the compound depreciation calculations is that while the interest adds value to the principal amount, the depreciation amount reduces value!

Worked Example 98: Compound Depreciation

Question: The Flamingo population of the Bergriver mouth is depreciating on a reducing balance at a rate of 12% p.a. If there is now 3 200 flamingos in the wetlands of the Bergriver mouth, how many will there be in 5 years' time ? Answer to three significant numbers.

Answer

Step 1 : Determine what has been provided and what is required

$$\begin{aligned}P &= R3\ 200 \\ i &= 0{,}12 \\ n &= 5 \\ A &\ is\ required\end{aligned}$$

Step 2 : Determine how to approach the problem

$$A = 3\ 200(1-0{,}12)^5$$

Step 3 : Solve the problem

$$\begin{aligned}A &= 3\ 200(0{,}88)^5 \\ &= 3\ 200 \times 0{,}527731916 \\ &= 1688{,}742134\end{aligned}$$

Step 4 : Write the final answer

There would be approximately 1 690 flamingos in 5 years' time.

Worked Example 99: Compound Depreciation

Question: Farmer Brown buys a tractor for R250 000 and depreciates it by 20% per year using the compound depreciation method. What is the depreciated value of the tractor after 5 years ?

Answer

Step 1 : Determine what has been provided and what is required

$$\begin{array}{rcl} P &=& R250\ 000 \\ i &=& 0{,}2 \\ n &=& 5 \\ A & is & required \end{array}$$

Step 2 : Determine how to approach the problem

$$A = 250\ 000(1 - 0{,}2)^5$$

Step 3 : Solve the problem

$$\begin{aligned} A &= 250\ 000(0{,}8)^5 \\ &= 250\ 000 \times 0{,}32768 \\ &= 81\ 920 \end{aligned}$$

Step 4 : Write the final answer
Depreciated value after 5 years is R 81 920

?

Exercise: Compound Depreciation

1. On January 1, 2008 the value of my Kia Sorento is R320 000. Each year after that, the cars value will decrease 20% of the previous years value. What is the value of the car on January 1, 2012.
2. The population of Bonduel decreases at a rate of 9,5% per annum as people migrate to the cities. Calculate the decrease in population over a period of 5 years if the initial population was 2 178 000.
3. A 20 kg watermelon consists of 98% water. If it is left outside in the sun it loses 3% of its water each day. How much does in weigh after a month of 31 days ?
4. A computer depreciates at $x\%$ per annum using the reducing-balance method. Four years ago the value of the computer was R10 000 and is now worth R4 520. Calculate the value of x correct to two decimal places.

21.5 Present Values or Future Values of an Investment or Loan

21.5.1 Now or Later

When we studied simple and compound interest we looked at having a sum of money now, and calculating what it will be worth in the future. Whether the money was borrowed or invested, the calculations examined what the total money would be at some future date. We call these *future values*.

It is also possible, however, to look at a sum of money in the future, and work out what it is worth now. This is called a *present value*.

For example, if R1 000 is deposited into a bank account now, the future value is what that amount will accrue to by some specified future date. However, if R1 000 is needed at some future time, then the present value can be found by working backwards - in other words, how much must be invested to ensure the money grows to R1 000 at that future date?

The equation we have been using so far in compound interest, which relates the open balance (P), the closing balance (A), the interest rate (i as a rate per annum) and the term (n in years) is:

$$A = P \cdot (1+i)^n \tag{21.2}$$

Using simple algebra, we can solve for P instead of A, and come up with:

$$P = A \cdot (1+i)^{-n} \tag{21.3}$$

This can also be written as follows, but the first approach is usually preferred.

$$P = A/(1+i)^n \tag{21.4}$$

Now think about what is happening here. In Equation 21.2, we start off with a sum of money and we let it grow for n years. In Equation 21.3 we have a sum of money which we know in n years time, and we "unwind" the interest - in other words we take off interest for n years, until we see what it is worth right now.

We can test this as follows. If I have R1 000 now and I invest it at 10% for 5 years, I will have:

$$\begin{aligned} A &= P \cdot (1+i)^n \\ &= \text{R1 000}(1+10\%)^5 \\ &= \text{R1 610,51} \end{aligned}$$

at the end. BUT, if I know I have to have R1 610,51 in 5 years time, I need to invest:

$$\begin{aligned} P &= A \cdot (1+i)^{-n} \\ &= \text{R1 610,51}(1+10\%)^{-5} \\ &= \text{R1 000} \end{aligned}$$

We end up with R1 000 which - if you think about it for a moment - is what we started off with. Do you see that?

Of course we could apply the same techniques to calculate a present value amount under simple interest rate assumptions - we just need to solve for the opening balance using the equations for simple interest.

$$A = P(1 + i \times n) \tag{21.5}$$

Solving for P gives:

$$P = A/(1 + i \times n) \tag{21.6}$$

Let us say you need to accumulate an amount of R1 210 in 3 years time, and a bank account pays *Simple Interest* of 7%. How much would you need to invest in this bank account today?

$$\begin{aligned} P &= \frac{A}{1+n \cdot i} \\ &= \frac{\text{R1 210}}{1+3 \times 7\%} \\ &= \text{R1 000} \end{aligned}$$

Does this look familiar? Look back to the simple interest worked example in Grade 10. There we started with an amount of R1 000 and looked at what it would grow to in 3 years' time using simple interest rates. Now we have worked backwards to see what amount we need as an opening balance in order to achieve the closing balance of R1 210.

In practice, however, present values are usually always calculated assuming compound interest. So unless you are explicitly asked to calculate a present value (or opening balance) using simple interest rates, make sure you use the compound interest rate formula!

Exercise: Present and Future Values

1. After a 20-year period Josh's lump sum investment matures to an amount of R313 550. How much did he invest if his money earned interest at a rate of 13,65% p.a.compounded half yearly for the first 10 years, 8,4% p.a. compounded quarterly for the next five years and 7,2% p.a. compounded monthly for the remaining period ?
2. A loan has to be returned in two equal semi-annual instalments. If the rate of interest is 16% per annum, compounded semi-annually and each instalment is R1 458, find the sum borrowed.

21.6 Finding i

By this stage in your studies of the mathematics of finance, you have always known what interest rate to use in the calculations, and how long the investment or loan will last. You have then either taken a known starting point and calculated a future value, or taken a known future value and calculated a present value.

But here are other questions you might ask:

1. I want to borrow R2 500 from my neighbour, who said I could pay back R3 000 in 8 months time. What interest is she charging me?
2. I will need R450 for some university textbooks in 1,5 years time. I currently have R400. What interest rate do I need to earn to meet this goal?

Each time that you see something different from what you have seen before, start off with the basic equation that you should recognise very well:

$$A = P \cdot (1+i)^n$$

If this were an algebra problem, and you were told to "solve for i", you should be able to show that:

$$\begin{aligned} A/P &= (1+i)^n \\ (1+i) &= (A/P)^{1/n} \\ i &= (A/P)^{1/n} - 1 \end{aligned}$$

You do not need to memorise this equation, it is easy to derive any time you need it!

So let us look at the two examples mentioned above.

1. Check that you agree that P=R2 500, A=R3 000, n=8/12=0,666667. This means that:

$$\begin{aligned} i &= (\text{R3 000/R2 500})^{1/0{,}666667} - 1 \\ &= 31{,}45\% \end{aligned}$$

Ouch! That is not a very generous neighbour you have.

2. Check that P=R400, A=R450, n=1,5

$$\begin{aligned} i &= (\text{R450}/\text{R400})^{1/1{,}5} - 1 \\ &= 8{,}17\% \end{aligned}$$

This means that as long as you can find a bank which pays more than 8,17% interest, you should have the money you need!

Note that in both examples, we expressed n as a number of years ($\frac{8}{12}$ years, not 8 because that is the number of months) which means i is the annual interest rate. Always keep this in mind - keep years with years to avoid making silly mistakes.

Exercise: Finding i

1. A machine costs R45 000 and has a scrap value of R9 000 after 10 years. Determine the annual rate of depreciation if it is calculated on the reducing balance method.
2. After 5 years an investment doubled in value. At what annual rate was interest compounded ?

21.7 Finding n - Trial and Error

By this stage you should be seeing a pattern. We have our standard formula, which has a number of variables:

$$A = P \cdot (1 + i)^n$$

We have solved for A (in section 8.5), P (in section 21.5) and i (in section 21.6). This time we are going to solve for n. In other words, if we know what the starting sum of money is and what it grows to, and if we know what interest rate applies - then we can work out how long the money needs to be invested for all those other numbers to tie up.

This section will calculate n by trial and error and by using a calculator. The proper algebraic solution will be learnt in Grade 12.

Solving for n, we can write:

$$\begin{aligned} A &= P(1 + i)^n \\ \frac{A}{P} &= (1 + i)^n \end{aligned}$$

Now we have to examine the numbers involved to try to determine what a possible value of n is. Refer to Table 5.1 (on page 38) for some ideas as to how to go about finding n.

Worked Example 100: Term of Investment - Trial and Error

Question: If we invest R3 500 into a savings account which pays 7,5% compound interest for an unknown period of time, at the end of which our account is worth R4 044,69. How long did we invest the money?

Answer

Step 1 : Determine what is given and what is required

- P=R3 500
- i=7,5%
- A=R4 044,69

We are required to find n.
Step 2 : Determine how to approach the problem
We know that:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{P} &= (1+i)^n \end{aligned}$$

Step 3 : Solve the problem

$$\begin{aligned} \frac{\text{R4 044,69}}{\text{R3 500}} &= (1+7{,}5\%)^n \\ 1{,}156 &= (1{,}075)^n \end{aligned}$$

We now use our calculator and try a few values for n.

Possible n	$1{,}075^n$
1,0	1,075
1,5	1,115
2,0	1,156
2,5	1,198

We see that n is close to 2.
Step 4 : Write final answer
The R3 500 was invested for about 2 years.

?

Exercise: Finding n - Trial and Error

1. A company buys two typs of motor cars: The Acura costs R80 600 and the Brata R101 700 VAT included. The Acura depreciates at a rate, compunded annually of 15,3% per year and the Brata at 19,7&, also compunded annually, per year. After how many years will the book value of the two models be the same ?
2. The fuel in the tank of a truck decreases every minute by 5,5% of the amount in the tank at that point in time. Calculate after how many minutes there will be less than $30l$ in the tank if it originally held $200l$.

21.8 Nominal and Effective Interest Rates

So far we have discussed annual interest rates, where the interest is quoted as a per annum amount. Although it has not been explicitly stated, we have assumed that when the interest is quoted as a per annum amount it means that the interest is once a year.

Interest however, may be paid more than just once a year, for example we could receive interest on a monthly basis, i.e. 12 times per year. So how do we compare a monthly interest rate, say, to an annual interest rate? This brings us to the concept of the effective annual interest rate.

One way to compare different rates and methods of interest payments would be to compare the Closing Balances under the different options, for a given Opening Balance. Another, more widely used, way is to calculate and compare the "effective annual interest rate" on each option. This way, regardless of the differences in how frequently the interest is paid, we can compare apples-with-apples.

For example, a savings account with an opening balance of R1 000 offers a compound interest rate of 1% per month which is paid at the end of every month. We can calculate the accumulated balance at the end of the year using the formulae from the previous section. But be careful our interest rate has been given as a monthly rate, so we need to use the same units (months) for our time period of measurement.

Remember, the trick to using the formulae is to define the time period, and use the interest rate relevant to the time period.

So we can calculate the amount that would be accumulated by the end of 1-year as follows:

$$\begin{aligned}\text{Closing Balance after 12 months} &= P \times (1+i)^n \\ &= \text{R1 000} \times (1+1\%)^{12} \\ &= \text{R1 126,83}\end{aligned}$$

Note that because we are using a monthly time period, we have used $n = 12$ months to calculate the balance at the end of one year.

The effective annual interest rate is an annual interest rate which represents the equivalent per annum interest rate assuming compounding.

It is the annual interest rate in our Compound Interest equation that equates to the same accumulated balance after one year. So we need to solve for the effective annual interest rate so that the accumulated balance is equal to our calculated amount of R1 126,83.

We use $i12$ to denote the monthly interest rate. We have introduced this notation here to distinguish between the annual interest rate, i. Specifically, we need to solve for i in the following equation:

$$\begin{aligned}P \times (1+i)^1 &= P \times (1+i12)^{12} \\ (1+i) &= (1+i12)^{12} \quad \text{divide both sides by } P \\ i &= (1+i12)^{12} - 1 \quad \text{subtract 1 from both sides}\end{aligned}$$

For the example, this means that the effective annual rate for a monthly rate $i12 = 1\%$ is:

$$\begin{aligned}i &= (1+i12)^{12} - 1 \\ &= (1+1\%)^{12} - 1 \\ &= 0{,}12683 \\ &= 12{,}683\%\end{aligned}$$

If we recalculate the closing balance using this annual rate we get:

$$\begin{aligned}\text{Closing Balance after 1 year} &= P \times (1+i)^n \\ &= \text{R1 000} \times (1+12{,}683\%)^1 \\ &= \text{R1 126,83}\end{aligned}$$

which is the same as the answer obtained for 12 months.

Note that this is greater than simply multiplying the monthly rate by 12 ($12 \times 1\% = 12\%$) due to the effects of compounding. The difference is due to interest on interest. We have seen this before, but it is an important point!

21.8.1 The General Formula

So we know how to convert a monthly interest rate into an effective annual interest. Similarly, we can convert a quarterly interest, or a semi-annual interest rate or an interest rate of any frequency for that matter into an effective annual interest rate.

For a quarterly interest rate of say 3% per quarter, the interest will be paid four times per year (every three month). We can calculate the effective annual interest rate by solving for i:

$$P(1+i) = P(1+i4)^4$$

where $i4$ is the quarterly interest rate.

So $(1+i) = (1{,}03)^4$, and so $i = 12{,}55\%$. This is the effective annual interest rate.

In general, for interest paid at a frequency of T times per annum, the follow equation holds:

$$P(1+i) = P(1+iT)^T \tag{21.7}$$

where iT is the interest rate paid T times per annum.

21.8.2 De-coding the Terminology

Market convention however, is not to state the interest rate as say 1% per month, but rather to express this amount as an annual amount which in this example would be paid monthly. This annual amount is called the nominal amount.

The market convention is to quote a nominal interest rate of "12% per annum paid monthly" instead of saying (an effective) 1% per month. We know from a previous example, that a nominal interest rate of 12% per annum paid monthly, equates to an effective annual interest rate of 12,68%, and the difference is due to the effects of interest-on-interest.

So if you are given an interest rate expressed as an annual rate but paid more frequently than annual, we first need to calculate the actual interest paid per period in order to calculate the effective annual interest rate.

$$\text{monthly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}} \tag{21.8}$$

For example, the monthly interest rate on 12% interest per annum paid monthly, is:

$$\begin{aligned}\text{monthly interest rate} &= \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}} \\ &= \frac{12\%}{12 \text{ months}} \\ &= 1\% \text{ per month}\end{aligned}$$

The same principle apply to other frequencies of payment.

Worked Example 101: Nominal Interest Rate

Question: Consider a savings account which pays a nominal interest at 8% per annum, paid quarterly. Calculate (a) the interest amount that is paid each quarter, and (b) the effective annual interest rate.

Answer

Step 1 : Determine what is given and what is required

We are given that a savings account has a nominal interest rate of 8% paid quarterly. We are required to find:

- the quarterly interest rate, $i4$
- the effective annual interest rate, i

Step 2 : Determine how to approach the problem

We know that:

$$\text{quarterly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of quarters per year}}$$

and

$$P(1+i) = P(1+iT)^T$$

where T is 4 because there are 4 payments each year.

Step 3 : Calculate the monthly interest rate

$$\begin{aligned}\text{quarterly interest rate} &= \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}} \\ &= \frac{8\%}{4 \text{ quarters}} \\ &= 2\% \text{ per quarter}\end{aligned}$$

Step 4 : Calculate the effective annual interest rate

The effective annual interest rate (i) is calculated as:

$$\begin{aligned}(1+i) &= (1+i4)^4 \\ (1+i) &= (1+2\%)^4 \\ i &= (1+2\%)^4 - 1 \\ &= 8{,}24\%\end{aligned}$$

Step 5 : Write the final answer

The quarterly interest rate is 2% and the effective annual interest rate is 8,24%, for a nominal interest rate of 8% paid quarterly.

Worked Example 102: Nominal Interest Rate

Question: On their saving accounts, Echo Bank offers an interest rate of 18% nominal, paid monthly. If you save R100 in such an account now, how much would the amount have accumulated to in 3 years' time?

Answer

Step 1 : Determine what is given and what is required

Interest rate is 18% nominal paid monthly. There are 12 months in a year. We are working with a yearly time period, so $n = 3$. The amount we have saved is $R100$, so $P = 100$. We need the accumulated value, A.

Step 2 : Recall relevant formulae

We know that

$$\text{monthly interest rate} = \frac{\text{Nominal interest Rate per annum}}{\text{number of periods per year}}$$

for converting from nominal interest rate to effective interest rate, we have

$$1+i = (1+iT)^T$$

and for cacluating accumulated value, we have

$$A = P \times (1+i)^n$$

Step 3 : Calculate the effective interest rate

There are 12 month in a year, so

$$\begin{aligned}i12 &= \frac{\text{Nominal annual interest rate}}{12} \\ &= \frac{18\%}{12} \\ &= 1{,}5\% \text{ per month}\end{aligned}$$

and then, we have

$$\begin{aligned} 1 + i &= (1 + i12)^{1}2 \\ i &= (1 + i12)^{1}2 - 1 \\ &= (1 + 1{,}5\%)^{1}2 - 1 \\ &= (1{,}015)^{1}2 - 1 \\ &= 19{,}56\% \end{aligned}$$

Step 4 : Reach the final answer

$$\begin{aligned} A &= P \times (1 + i)^{n} \\ &= 100 \times (1 + 19{,}56\%)^{3} \\ &= 100 \times 1{,}7091 \\ &= 170{,}91 \end{aligned}$$

Step 5 : Write the final answer
The accumulated value is $R170{,}91$. (Remember to round off to the the nearest cent.)

Exercise: Nominal and Effect Interest Rates

1. Calculate the effective rate equivalent to a nominal interest rate of 8,75% p.a. compounded monthly.
2. Cebela is quoted a nominal interest rate of 9,15% per annum compounded every four months on her investment of R 85 000. Calculate the effective rate per annum.

21.9 Formulae Sheet

As an easy reference, here are the key formulae that we derived and used during this chapter. While memorising them is nice (there are not many), it is the application that is useful. Financial experts are not paid a salary in order to recite formulae, they are paid a salary to use the right methods to solve financial problems.

21.9.1 Definitions

P	Principal (the amount of money at the starting point of the calculation)
i	interest rate, normally the effective rate per annum
n	period for which the investment is made
iT	the interest rate paid T times per annum, i.e. $iT = \frac{\text{Nominal Interest Rate}}{T}$

21.9.2 Equations

$$\begin{aligned}
Simple\ Increase : A &= P(1 + i \times n) \\
Compound\ Increase : A &= P(1 + i)^n \\
Simple\ Decrease : A &= P(1 - i \times n) \\
Compound\ Decrease : A &= P(1 - i)^n \\
Effective\ Annual\ Interest\ Rate(i) : (1 + i) &= (1 + iT)^T
\end{aligned}$$

21.10 End of Chapter Exercises

1. Shrek buys a Mercedes worth R385 000 in 2007. What will the value of the Mercedes be at the end of 2013 if
   A the car depreciates at 6% p.a. straight-line depreciation
   B the car depreciates at 12% p.a. reducing-balance depreciation.
2. Greg enters into a 5-year hire-purchase agreement to buy a computer for R8 900. The interest rate is quoted as 11% per annum based on simple interest. Calculate the required monthly payment for this contract.
3. A computer is purchased for R16 000. It depreciates at 15% per annum.
   A Determine the book value of the computer after 3 years if depreciation is calculated according to the straight-line method.
   B Find the rate, according to the reducing-balance method, that would yield the same book value as in 3a after 3 years.
4. Maggie invests R12 500,00 for 5 years at 12% per annum compounded monthly for the first 2 years and 14% per annum compounded semi-annually for the next 3 years. How much will Maggie receive in total after 5 years?
5. Tintin invests R120 000. He is quoted a nominal interest rate of 7,2% per annum compounded monthly.
   A Calculate the effective rate per annum correct to THREE decimal places.
   B Use the effective rate to calculate the value of Tintin's investment if he invested the money for 3 years.
   C Suppose Tintin invests his money for a total period of 4 years, but after 18 months makes a withdrawal of R20 000, how much will he receive at the end of the 4 years?
6. Paris opens accounts at a number of clothing stores and spends freely. She gets heself into terrible debt and she cannot pay off her accounts. She owes Hilton Fashion world R5 000 and the shop agrees to let Paris pay the bill at a nominal interest rate of 24% compounded monthly.
   A How much money will she owe Hilton Fashion World after two years ?
   B What is the effective rate of interest that Hilton Fashion World is charging her ?

Appendix A

GNU Free Documentation License

PREAMBLE

The purpose of this License is to make a manual, textbook, or other functional and useful document "free" in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or non-commercially. Secondarily, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of "copyleft", which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The "Document", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "you". You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A "Modified Version" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A "Secondary Section" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The “Invariant Sections” are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The “Cover Texts” are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A “Transparent” copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not “Transparent” is called “Opaque”.

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The “Title Page” means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, “Title Page” means the text near the most prominent appearance of the work’s title, preceding the beginning of the body of the text.

A section “Entitled XYZ” means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as “Acknowledgements”, “Dedications”, “Endorsements”, or “History”.) To “Preserve the Title” of such a section when you modify the Document means that it remains a section “Entitled XYZ” according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or non-commercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section A.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document’s license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections A and A above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

1. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
2. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
3. State on the Title page the name of the publisher of the Modified Version, as the publisher.
4. Preserve all the copyright notices of the Document.
5. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
6. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
7. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
8. Include an unaltered copy of this License.
9. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.

10. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.

11. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.

12. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.

13. Delete any section Entitled "Endorsements". Such a section may not be included in the Modified Version.

14. Do not re-title any existing section to be Entitled "Endorsements" or to conflict in title with any Invariant Section.

15. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section Entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties–for example, statements of peer review or that the text has been approved by an organisation as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section A above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled "History" in the various original documents, forming one section Entitled "History"; likewise combine any sections Entitled "Acknowledgements", and any sections Entitled "Dedications". You must delete all sections Entitled "Endorsements".

COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an "aggregate" if the copyright resulting from the compilation is not used to limit the legal rights of the compilation's users beyond what the individual works permit. When the Document is included an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section A is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document's Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section A. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled "Acknowledgements", "Dedications", or "History", the requirement (section A) to Preserve its Title (section A) will typically require changing the actual title.

TERMINATION

You may not copy, modify, sub-license, or distribute the Document except as expressly provided for under this License. Any other attempt to copy, modify, sub-license or distribute the Document is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See http://www.gnu.org/copyleft/.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

> Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the "with...Texts." line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.